OCR
RECOGNISING ACHIEVEMENT

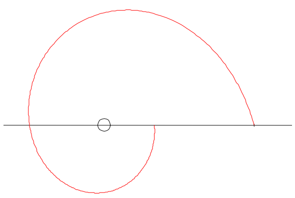
GCE

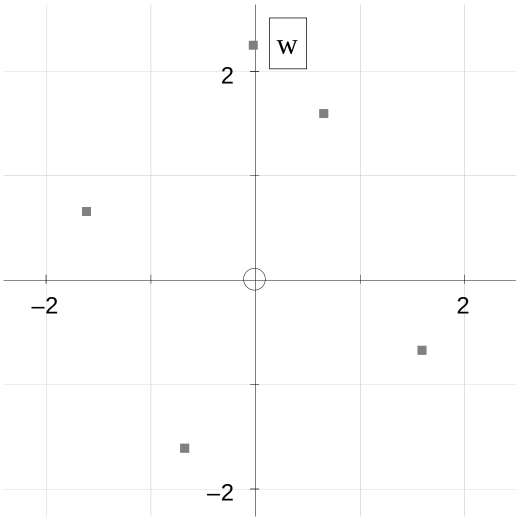
Mathematics (MEI)

Advanced GCE **4756**

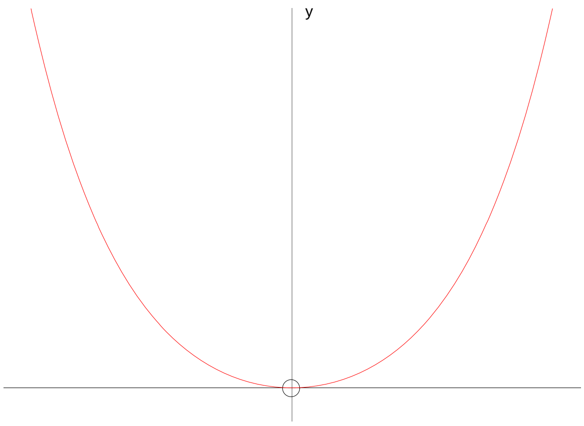
Further Methods for Advanced Mathematics (FP2)

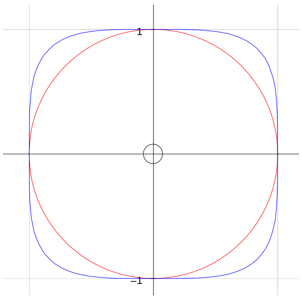
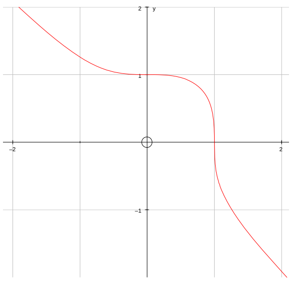
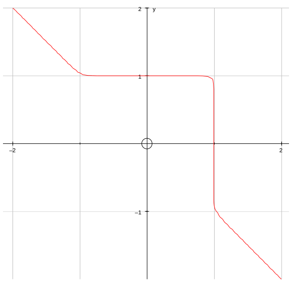
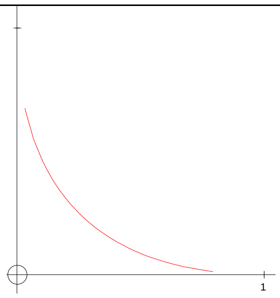
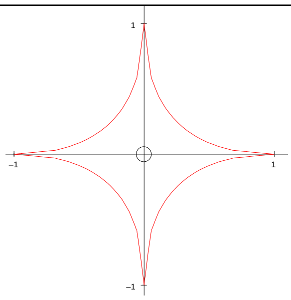
Mark Scheme for June 2010

1 (a)(i)	$f(t) = \arcsin t$ $\Rightarrow f'(t) = \dfrac{1}{\sqrt{1-t^2}} = \left(1-t^2\right)^{-\frac{1}{2}}$ $\Rightarrow f''(t) = -\frac{1}{2}\left(1-t^2\right)^{-\frac{3}{2}} \times -2t$ $= \dfrac{t}{\left(1-t^2\right)^{\frac{3}{2}}}$	B1 M1 A1 (ag) **3**	Any form Using Chain Rule
(ii)	$f(x) = \arcsin(x + ½)$ $\Rightarrow f(0) = \arcsin(½) = \dfrac{\pi}{6}$ $f'(0) = \left(1-\left(\frac{1}{2}\right)^2\right)^{-\frac{1}{2}} = \dfrac{2}{\sqrt{3}}$ and $f''(0) = \dfrac{\frac{1}{2}}{\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{3}{2}}} = \dfrac{4\sqrt{3}}{9}$ $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2} f''(0) + \ldots$ $\Rightarrow$ term in x^2 is $\dfrac{2\sqrt{3}}{9}x^2$	B1 (ag) M1 A1 (ag) M1 A1 **5**	$\dfrac{\pi}{6}$ obtained clearly from $f(0)$ www Clear substitution of $x = 0$ or $t = ½$ Evaluating $f''(0)$ and dividing by 2 Accept $0.385x^2$ or better
(b)	 Area $= \displaystyle\int_0^{\pi} \frac{1}{2} r^2 \, d\theta$ $= \displaystyle\int_0^{\pi} \frac{\pi^2 a^2}{2(\pi+\theta)^2} d\theta = \frac{\pi^2 a^2}{2}\int_0^{\pi} \frac{1}{(\pi+\theta)^2} d\theta$ $= \dfrac{\pi^2 a^2}{2}\left[\dfrac{-1}{\pi+\theta}\right]_0^{\pi}$ $= \dfrac{\pi^2 a^2}{2}\left(\dfrac{-1}{2\pi} + \dfrac{1}{\pi}\right)$ $= \frac{1}{4}\pi a^2$	G1 G1 M1 A1 M1 A1 **6**	Complete spiral with $r(2\pi) < r(0)$ $r(0) = a$, $r(2\pi) = a/3$ indicated or $r(0) > r(\pi/2) > r(\pi) > r(3\pi/2) > r(2\pi)$ Dep. on G1 above Max. G1 if not fully correct Integral expression involving r^2 Correct result of integration with correct limits Substituting limits into an expression of the form $\dfrac{k}{\pi+\theta}$. Dep. on M1 above
(c)	$\displaystyle\int_0^{\frac{3}{2}} \frac{1}{9+4x^2} dx = \frac{1}{4}\int_0^{\frac{3}{2}} \frac{1}{\frac{9}{4}+x^2} dx = \frac{1}{4} \times \left[\frac{2}{3}\arctan\frac{2x}{3}\right]_0^{\frac{3}{2}}$ $= \dfrac{1}{6}\arctan 1$ $= \dfrac{\pi}{24}$	M1 A1A1 M1 A1 **5**	arctan $\dfrac{1}{4} \times \dfrac{2}{3}$ and $\dfrac{2x}{3}$ Substituting limits. Dep. on M1 above Evaluated in terms of π **19**

2 (a)	$z^n + \frac{1}{z^n} = 2\cos n\theta$, $z^n - \frac{1}{z^n} = 2j\sin n\theta$	B1	Both
	$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$	M1	Expanding $\left(z - \frac{1}{z}\right)^5$
	$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$		
	$\Rightarrow$ $32j\sin^5\theta = 2j\sin 5\theta - 10j\sin 3\theta + 20j\sin\theta$	M1 A1	Introducing sines (and possibly cosines) of multiple angles RHS
	$\Rightarrow$ $\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	A1ft	Division by 32(j)
	$A = \frac{5}{8}$, $B = -\frac{5}{16}$, $C = \frac{1}{16}$	**5**	
(b)(i)	4th roots of $-9j = 9e^{\frac{3}{2}\pi j}$ are $re^{j\theta}$ where		
	$r = \sqrt{3}$	B1	Accept $9^{\frac{1}{4}}$
	$\theta = \frac{3\pi}{8}$	B1	
	$\Rightarrow$ $\theta = \frac{3\pi}{8} + \frac{2k\pi}{4}$	M1	Implied by at least two correct (ft) further values
	$\Rightarrow$ $\theta = \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$	A1	Or stating k = (0), 1, 2, 3 Allow arguments in range $-\pi \le \theta \le \pi$
		M1 A1 **6**	Points at vertices of a square centre O or 3 correct points (ft) or 1 point in each quadrant
(ii)	Mid-point of SP has argument $\frac{\pi}{8}$	B1	
	and modulus of $\sqrt{\frac{3}{2}}$	B1	
	Argument of $w = 4 \times \frac{\pi}{8} = \frac{\pi}{2}$		
	and modulus $= \left(\sqrt{\frac{3}{2}}\right)^4 = \frac{9}{4}$	M1 A1 G1 **5**	Multiplying argument by 4 and modulus raised to power of 4 Both correct w plotted on imag. axis above level of P **16**

3 (a)(i)	$2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0 \Rightarrow (\lambda - 2)(2\lambda^2 + 5\lambda - 3) = 0$ $\Rightarrow \lambda = 2$ or $2\lambda^2 + 5\lambda - 3 = 0$ $\Rightarrow (2\lambda - 1)(\lambda + 3) = 0$ $\Rightarrow \lambda = ½, \lambda = -3$	B1 M1 A1A1 **4**	Substituting $\lambda = 2$ or factorising Obtaining and solving a quadratic
(ii)	$\mathbf{M}\begin{pmatrix}3\\-3\\1\end{pmatrix} = 2\begin{pmatrix}3\\-3\\1\end{pmatrix} = \begin{pmatrix}6\\-6\\2\end{pmatrix}$	B1	
	$\mathbf{M}^2\mathbf{v} = 2^2\mathbf{v} = 4\begin{pmatrix}1\\-1\\\frac{1}{3}\end{pmatrix} = \begin{pmatrix}4\\-4\\\frac{4}{3}\end{pmatrix}$	B2	Give B1 for one component with the wrong sign
	$\mathbf{M}\begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\end{pmatrix} = 2\begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}3\\-3\\1\end{pmatrix}$	M1	Recognising that the solution is a multiple of the given RHS
	$\Rightarrow x = \frac{3}{2}, y = -\frac{3}{2}, z = \frac{1}{2}$	A1 **5**	Correct multiple
(iii)	$2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0$ $\Rightarrow 2\mathbf{M}^3 + \mathbf{M}^2 - 13\mathbf{M} + 6\mathbf{I} = \mathbf{0}$ $\Rightarrow \mathbf{M}^3 = -\frac{1}{2}\mathbf{M}^2 + \frac{13}{2}\mathbf{M} - 3\mathbf{I}$ $\Rightarrow \mathbf{M}^4 = -\frac{1}{2}\mathbf{M}^3 + \frac{13}{2}\mathbf{M}^2 - 3\mathbf{M}$ $\Rightarrow \mathbf{M}^4 = -\frac{1}{2}(-\frac{1}{2}\mathbf{M}^2 + \frac{13}{2}\mathbf{M} - 3\mathbf{I}) + \frac{13}{2}\mathbf{M}^2 - 3\mathbf{M}$ $\Rightarrow \mathbf{M}^4 = \frac{27}{4}\mathbf{M}^2 - \frac{25}{4}\mathbf{M} + \frac{3}{2}\mathbf{I}$ $A = \frac{27}{4}, B = -\frac{25}{4}, C = \frac{3}{2}$	 M1 M1 M1 A1 **4**	 Using Cayley-Hamilton Theorem Multiplying by **M** Substituting for $\mathbf{M}^3$
(b)	$\mathbf{N} = \mathbf{PDP}^{-1}$	B1	Order must be correct
	where $\mathbf{D} = \begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}$	B1	
	and $\mathbf{P} = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}$	B1	For B1B1, order must be consistent
	$\Rightarrow \mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$	B1ft	Ft their **P**
	$\Rightarrow \mathbf{N} = \frac{1}{3}\begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$		
	$= \frac{1}{3}\begin{pmatrix}-1 & -2\\-2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$	M1	Attempting matrix product
	$= \frac{1}{3}\begin{pmatrix}3 & -3\\-6 & 0\end{pmatrix} = \begin{pmatrix}1 & -1\\-2 & 0\end{pmatrix}$	A1	
	OR Let $\mathbf{N} = \begin{pmatrix}a & c\\b & d\end{pmatrix}$		
	$\begin{pmatrix}a & c\\b & d\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = -1\begin{pmatrix}1\\2\end{pmatrix}$ B1		Or $\begin{pmatrix}a+1 & c\\b & d+1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$
	$\begin{pmatrix}a & c\\b & d\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 2\begin{pmatrix}-1\\1\end{pmatrix}$ B1		Or $\begin{pmatrix}a-2 & c\\b & d-2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$
	$\Rightarrow a + 2c = -1, -a + c = -2$ B1 $b + 2d = -2, -b + d = 2$ B1 $\Rightarrow a = 1, c = -1; b = -2, d = 0$ M1A1		 Solving both pairs of equations
		6	**19**

4 (i)	$2\sinh x\cosh x$ $= 2 \times \dfrac{e^x + e^{-x}}{2} \times \dfrac{e^x - e^{-x}}{2}$ $= \dfrac{e^{2x} - e^{-2x}}{2}$ $= \sinh 2x$ Differentiating, $2\cosh 2x = 2\cosh^2 x + 2\sinh^2 x$ $\Rightarrow \cosh 2x = \cosh^2 x + \sinh^2 x$	M1 A1 (ag) B1 B1 **4**	Using exponential definitions and multiplying or factorising One side correct Correct completion
(ii)	 Volume $= \pi\int_0^2 (\cosh x - 1)^2\, dx$ $= \pi\int_0^2 \cosh^2 x - 2\cosh x + 1\, dx$ $= \pi\int_0^2 \tfrac{1}{2}\cosh 2x - 2\cosh x + \tfrac{3}{2}\, dx$ $= \pi\left[\tfrac{1}{4}\sinh 2x - 2\sinh x + \tfrac{3}{2}x\right]_0^2$ $= \pi\left[\tfrac{1}{4}\sinh 4 - 2\sinh 2 + 3\right]$ $= 8.070$	G1 M1 A1 M1 A2 A1 **7**	Correct shape and through origin $\int (\cosh x - 1)^2\, dx$ A correct expanded integral expression including limits 0, 2 (may be implied by later work) Attempting to obtain an integrable form Dep. on M1 above Give A1 for two terms correct 3 d.p. required. Condone 8.07
(iii)	$y = \cosh 2x + \sinh x$ $\Rightarrow \dfrac{dy}{dx} = 2\sinh 2x + \cosh x$ At S.P. $2\sinh 2x + \cosh x = 0$ $\Rightarrow 4\sinh x\cosh x + \cosh x = 0$ $\Rightarrow \cosh x(4\sinh x + 1) = 0$ $\Rightarrow \cosh x = 0$ (rejected) $\Rightarrow \sinh x = -\tfrac{1}{4}$ $\Rightarrow x = \ln\left(-\dfrac{1}{4} + \dfrac{\sqrt{17}}{4}\right)$	B1 M1 M1 A1 A1 M1 A1 **7**	Any correct form Setting derivative equal to zero and using identity Solving $\dfrac{dy}{dx}$ =0 to obtain value of sinh x Repudiating cosh x = 0 Using log form of arsinh, or setting up and solving quadratic in e^x A0 if extra "roots" quoted **18**

5(i)(A)	Circle	B1	
(B)		G1 G1	Sketch of circle, centre (0, 0) Sketch of "squarer" circle on same axes
(C)	Square	B1	
(D)	$-1 \le x \le 1$ $-1 \le y \le 1$	B1 B1 **6**	Give B1B0 for not all non-strict or unclear
(ii)(A)	Odd roots exist for all real numbers	B1	Any equivalent explanation
(B)	Line	B1	Sketch insufficient
(C)	 Asymptote: $x + y = 0$	G1 B1	
(D)		G1 G1 **6**	Line $x + y = 0$ outside unit square Lines $y = 1$ and $x = 1$ on unit square
(iii)	 $0 \le x, y \le 1$	G1 B1 **2**	G0 if curve beyond (1, 0) or (0, 1) Accept strict, or indication on graph
(iv)(A)		G2ft	Give G1 for a partial attempt. Ft from (iii) on shape
(B)	Limit is a "plus sign" where $x \to 0$ for $-1 \le y \le 1$ and vice versa	B1 B1 **4**	**18**